Inequalities for general mixed affine surface areas *

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Abstract

Several general mixed affine surface areas are introduced. We prove some important properties, such as, affine invariance, for these general mixed affine surface areas. We also establish new Alexandrov-Fenchel type inequalities, Santaló-type inequalities, and affine isoperimetric inequalities for these general mixed affine surface areas.

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1 Introduction

There has been a growing body of work in isoperimetric inequalities. The classical isoperimetric inequality, which compares the surface area in terms of the volume, is an extremely powerful tool in geometry and related areas. Relatively more important results in the family of isoperimetric inequalities, e.g., the celebrate Blaschke-Santaló inequality, have the "affine invariant" flavor. These affine isoperimetric inequalities compare two functionals associated with convex bodies (or more general sets) where the ratio of the functionals is invariant under non-degenerate linear transformations. Important functionals include but are not limited to, the volume, L_p affine surface areas, mixed p-affine surface areas, and general affine surface areas.

The study of affine surface areas has a long history. The notion of the classical affine surface area was first introduced by Blaschke in 1923 [6], and was first generalized to the L_p affine surface area for $p > 1$ by Lutwak in [31]. Since then, considerable attention has been paid to the L_p affine surface area, which is now at

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the core of the rapidly developing L_p-Brunn-Minkowski theory [10, 11, 13, 19, 23, 26, 30, 33, 36, 47, 48, 49] among others. The L_p affine surface area was further extended to all $p \in \mathbb{R}$ via geometric interpretations [37, 45, 46, 51]. For a sufficiently smooth convex body K in $\mathbb{R}^n$, the L_p affine surface area $as_p(K)$ of K was defined as in [31] ($p > 1$) and [46] ($p < 1$) by

$$as_p(K) = \int_{S^{n-1}} \Big[h_K(u)^{1-p} f_K(u)\Big]^{\frac{n}{n+p}} \, d\sigma(u).$$

Here S^{n-1} is the boundary of the unit Euclidean ball B_2^n in $\mathbb{R}^n$, σ is the usual surface area measure on S^{n-1}, $h_K(u)$ is the support function of the convex body K at $u \in S^{n-1}$, and $f_K(u)$ is the curvature function of K at u, i.e., the reciprocal of the Gauss curvature $\kappa_K(x)$ at the point $x \in \partial K$, the boundary of K, that has u as its outer normal. The L_p affine surface area is the key ingredient in many problems, such as, approximation of convex bodies by polytopes [16, 27, 46], theory of valuation (see e.g. [2, 3, 21, 25]), and the L_p affine isoperimetric inequality [31, 52]. Recently, Paouris and Werner [40] linked the L_p affine surface area with the relative entropy of cone measures of K and of its polar $K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \forall x \in K\}$, where $\langle x, y\rangle$ is the inner product of x and y.

In literature, two generalizations of the L_p affine surface area are important: the mixed p-affine surface area [29, 31, 50, 53] and the general affine surface areas by Ludwig [24, 26]. The mixed p-affine surface area, which involves n convex bodies in $\mathbb{R}^n$, takes the form

$$as_p(K_1, \cdots, K_n) = \int_{S^{n-1}} \Big[h_{K_1}(u)^{1-p} f_{K_1}(u) \cdots h_{K_n}(u)^{1-p} f_{K_n}(u)\Big]^{\frac{1}{n+p}} \, d\sigma(u).$$

Clearly, $as_p(K) = as_p(K, \cdots, K)$ if all $K_i = K$. Moreover, the mixed p-affine surface area contains many other important functionals of convex bodies as special cases, such as, the usual surface area and the dual mixed volume [28]. The discovery of the general affine surface areas owes to the valuation theory [26]. These general affine surface areas involve very general functions, for instance, the L_ϕ-affine surface area associated with $\phi \in Conc(0, \infty)$ has the form

$$as_\phi(K) \;\; = \int_{S^{n-1}} \phi\bigg(\frac{1}{f_K(u) h_K^{n+1}(u)}\bigg) h_K(u) f_K(u) \, d\sigma(u).$$

When $\phi(t) = t^{\frac{p}{n+p}}$ for $p > 0$, $as_\phi(K) = as_p(K)$. A fundamental result on the L_ϕ-affine surface area is the characterization theory of upper-semicontinuous $SL(n)$

invariant valuation [26], that is, every upper-semicontinuous, $SL(n)$ invariant valuation vanishing on polytopes can be represented as a L_ϕ-affine surface area for some $\phi \in Conc(0,\infty)$. We refer readers to [24, 26] for other general affine surfaces and their properties.

In this paper, we introduce several general mixed affine surface areas, which are the combinations of the above two generalizations. Throughout the whole paper, $K \in C^2_+$ means that K has the origin in its interior, and has C^2 boundary with everywhere strictly positive Gaussian curvature. Hereafter, let $Conc(0,\infty)$ be the set of functions $\phi:(0,\infty)\rightarrow(0,\infty)$ such that either ϕ is a nonzero constant function, or ϕ is concave with $\lim_{t\rightarrow 0}\phi(t)=0$ and $\lim_{t\rightarrow\infty}\phi(t)/t=0$ (in this case, we set $\phi(0)=0$). For all $\phi_i \in Conc(0,\infty)$ and all $K_i \in C^2_+$, we define the general mixed L_ϕ-affine surface area by

$$as(\phi_1, K_1;\cdots;\phi_n, K_n)=\int_{S^{n-1}}\prod_{i=1}^{n}\left[\phi_i\left(\frac{1}{f_{K_i}(u)h_{K_i}^{n+1}(u)}\right)h_{K_i}(u)f_{K_i}(u)\right]^{\frac{1}{n}}d\sigma(u). \qquad (1.1)$$

Clearly, $as(\phi,K;\cdots;\phi,K) = as_\phi(K)$, and $as(\phi,K_1;\cdots;\phi,K_n) = as_p(K_1,\cdots,K_n)$ if $\phi(t)=t^{\frac{p}{n+p}}$ for $p\geq 0$. Hence, we include the L_p affine surface area for $p>0$ and the volume (i.e., for $p = 0$) as special cases. We show that the general mixed L_ϕ-affine surface area is affine invariant for all $\phi_i \in Conc(0,\infty)$ (see Theorem 2.2), and also provide geometric interpretations of it (see Theorem 2.1). See Section 2 for other general mixed affine surface areas and their properties, in particular, we prove a duality result (Proposition 2.2).

The L_p affine isoperimetric inequality for the L_p affine surface area with all $p \in \mathbb{R}$ was proved in [52] (see [31] for $p \geq 1$). The L_p affine isoperimetric inequalities are fundamental in many problems and have various applications in, e.g., imaging recognition and PDE (e.g. [14, 15, 32, 44]). In particular, it was used by Andrews [4, 5], Sapiro and Tannenbaum [43] to show the uniqueness of self-similar solutions of the affine curvature flow and to study its asymptotic behavior. Other related works include, e.g., [9, 17, 34, 35]. The L_p affine isoperimetric inequalities were extended to the mixed p-affine surface areas [31, 53] and the general affine surface areas [24]. In this paper, we prove analogous affine isoperimetric inequalities for general mixed affine surface areas. Let B_{K_i} be the origin-symmetric (Euclidean) ball with $|B_{K_i}| = |K_i|$ for all i.

Theorem 3.2 *Let all $K_i \in C^2_+$ be convex bodies with centroid at the origin.*

(i): If all $\phi_i \in Conc(0,\infty)$*, then*

$$as(\phi_1, K_1; \cdots ; \phi_n, K_n) \le as(\phi_1, B_{K_1}; \cdots ; \phi_n, B_{K_n}).$$

Equality holds if all K_i *are ellipsoids that are dilates of one another.*

(ii) For all $\phi_i \in Conc(0,\infty)$ *with homogeneous degrees* $r_i \in [0,1)$*,*

$$\left(\frac{as(\phi_1, K_1; \cdots ; \phi_n, K_n)}{as(\phi_1, B_2^n; \cdots ; \phi_n, B_2^n)}\right)^n \le \prod_{i=1}^n \left(\frac{|K_i|}{|B_2^n|}\right)^{1-2r_i},$$

with equality if all K_i *are ellipsoids that are dilates of one another.*

Blaschke-Santaló inequality and the inverse Santaló inequality are model examples of affine isoperimetric inequalities. Blaschke-Santaló inequality was proved by the L_1 affine isoperimetric inequality in [42] (where the Santaló point instead of the centroid was used). Note that if K has Santaló point at the origin, then its polar body K° has centroid at the origin (see [44] for details). The inverse Santaló inequality was due to Bourgain and Milman [7] (see also [22, 38, 39, 41]). They were successfully extended to affine surface areas in [31, 50, 52, 53] among others. Here we prove the following Santaló-type inequalities for the general mixed L_ϕ-affine surface areas.

Theorem 3.1 *Let* $K_i \in C_+^2$ *and* $\phi_i \in Conc(0,\infty)$ *be homogeneous of degrees* $r_i \in [0,1)$*. Then*

$$as(\phi_1, K_1; \cdots ; \phi_n, K_n) as(\phi_1, K_1^\circ; \cdots ; \phi_n, K_n^\circ) \le n^2 \prod_{i=1}^n \left[\phi_i(1)^2 \; |K_i| \; |K_i^\circ|\right]^{\frac{1}{n}}.$$

Moreover, if all $K_i \in C_+^2$ *have centroid at the origin, then*

$$as(\phi_1, K_1; \cdots ; \phi_n, K_n) as(\phi_1, K_1^\circ; \cdots ; \phi_n, K_n^\circ) \le \left[as(\phi_1, B_2^n; \cdots ; \phi_n, B_2^n)\right]^2,$$

with equality if all K_i *are origin-symmetric ellipsoids that are dilates of one another.*

Another powerful inequality in geometry is the well-known classical Alexandrov-Fenchel inequality for mixed volume (see [1, 8, 44]). Here we show Alexandrov-Fenchel type inequalities for the general mixed affine surface areas, which have similar forms to the classical Alexandrov-Fenchel inequality. However, they do not imply the classical Alexandrov-Fenchel inequality. See also [28, 29, 53] for more such type inequalities.

Proposition 3.1 *Let all $K_i \in C^2_+$ and $\phi_i \in Conc(0,\infty)$. Then, for $1 \le m \le n$,*

$$as^m(\phi_1,K_1;\cdots;\phi_n,K_n) \le \prod_{i=0}^{m-1} as(\phi_1,K_1;\cdots;\phi_{n-m},K_{n-m};\underbrace{\phi_{n-i},K_{n-i};\cdots;\phi_{n-i},K_{n-i}}_{m}).$$

Equality holds if (1) all K_i coincide and $\phi_i = \lambda_i\phi_n$ for some $\lambda_i > 0$, $i = n-m+1,\cdots,n$, or (2) $\phi_i = \lambda_i\phi_n$, $K_i = \eta_i K_n$ for some $\lambda_i, \eta_i > 0$, $i = n-m+1,\cdots,n$, and ϕ_n is homogeneous of degree $r \in [0,1)$. The equality holds trivially if $m = 1$.

In particular, $as^n(\phi_1,K_1;\cdots;\phi_n,K_n) \le as_{\phi_1}(K_1)\cdots as_{\phi_n}(K_n)$.

This paper is organized as follows. In Section 2, we introduce several general mixed affine surface areas. We provide geometric interpretations of them and prove some important properties of them, such as, affine invariant properties. In Section 3, we establish new Alexandrov-Fenchel type inequalities, Santaló-type inequalities, and affine isoperimetric inequalities for these general mixed affine surface areas. Section 4 dedicates to the general i-th mixed affine surface areas. Similar Santaló-type and affine isoperimetric inequalities are also proved.

2 General mixed affine surface areas

2.1 General mixed L_ϕ- and L_ψ-affine surface areas

The L_p affine surface area for $-n < p < 0$ is associated with the L_ψ-affine surface area, where $\psi \in Conv(0,\infty)$ in [24]. Hereafter, $Conv(0,\infty)$ is the set of functions $\psi : (0,\infty) \to (0,\infty)$ such that either ψ is a nonzero constant function, or ψ is convex with $\lim_{t\to 0}\psi(t) = \infty$ and $\lim_{t\to\infty}\psi(t) = 0$ (in this case, we set $\psi(0) = \infty$). For $\psi_i \in Conv(0,\infty)$ and $K_i \in C^2_+$, we define the general mixed L_ψ-affine surface area as

$$as(\psi_1,K_1;\cdots;\psi_n,K_n) = \int_{S^{n-1}} \prod_{i=1}^{n} \left[\psi_i\left(\frac{1}{f_{K_i}(u)h_{K_i}^{n+1}(u)}\right) h_{K_i}(u) f_{K_i}(u)\right]^{\frac{1}{n}} d\sigma(u).$$

In particular, $as(\psi,K;\cdots;\psi,K) = as_\psi(K)$ is the L_ψ-affine surface area of K introduced in [24]. When all $\psi_i(t) = t^{\frac{p}{n+p}}$ for $-n < p \le 0$, one gets the mixed p-affine surface area for $-n < p \le 0$. In particular, one includes the L_p affine surface area for $-n < p < 0$ as a special case. If all $\psi_i = \psi$, then we use $as_\psi(K_1,\cdots,K_n)$ to represent $as(\psi,K_1;\cdots;\psi,K_n)$. We use $as(\psi_1,\cdots,\psi_n;K)$ instead of $as(\psi_1,K;\cdots;\psi_n,K)$ when all $K_i = K$. Clearly, $as(\psi_1,\cdots,\psi_n;B^n_2) = \left[\psi_1(1)\cdots\psi_n(1)\right]^{\frac{1}{n}} n|B^n_2|$.

We always assume that $\phi \in Conc(0,\infty)$ is nonzero. As above, if all $\phi_i = \phi$, we write $as_\phi(K_1,\cdots,K_n)$ for $as(\phi,K_1;\cdots;\phi,K_n)$. We use $as(\phi_1,\cdots,\phi_n;K)$ for $as(\phi_1,K;\cdots;\phi_n,K)$ if all $K_i = K$. Clearly, $as(\phi_1,\cdots,\phi_n;B_2^n) = \big[\phi_1(1)\cdots\phi_n(1)\big]^{\frac{1}{n}} n|B_2^n|$.

The following theorem gives a geometric interpretation for the general mixed L_ϕ-affine surface area by the illumination surface body. Similar geometric interpretations can also be obtained by the surface body [46, 52].

Definition 2.1 (**Illumination surface body**) [53] *Let $s \geq 0$ and $f: \partial K \to \mathbb{R}$ be a nonnegative, integrable function. The illumination surface body $K^{f,s}$ is defined as*

$$K^{f,s} = \left\{ x : \int_{\partial K \cap \overline{[x,K] \setminus K}} f \, d\mu_K \leq s \right\}.$$

Here, μ_K is the usual surface measure of ∂K, $[x, K]$ denotes the convex hull of x and K, $A \setminus B = \{z : z \in A,\ but\ z \notin B\}$, and $\bar{A}$ is the closure of A.

Theorem 2.1 *Let $K, K_i \in C_+^2$ and $\phi_i \in Conc(0,\infty)$, $i = 1, \cdots, n$. Let $f : \partial K \to \mathbb{R}$ be the function*

$$f(N_K^{-1}(u)) = f_K(u)^{\frac{n-2}{2}} \prod_{i=1}^{n} \left[\phi_i\left(\frac{1}{f_{K_i}(u) h_{K_i}^{n+1}(u)} \right) h_{K_i}(u) f_{K_i}(u) \right]^{\frac{1-n}{2n}}, \qquad (2.2)$$

where $N_K^{-1} : S^{n-1} \to \partial K$ is the inverse Gauss map, that is, $N_K^{-1}(N_K(x)) = x$ for all $x \in \partial K$. Let $c_n = 2|B_2^{n-1}|^{\frac{2}{n-1}}$. Then,

$$as(\phi_1, K_1; \cdots; \phi_n, K_n) = \lim_{s\to 0} c_n \; \frac{|K^{f,s}| - |K|}{s^{\frac{2}{n-1}}}.$$

Proof. Theorem 4.1 and its following remark in [53] imply that

$$\begin{aligned}
\lim_{s\to 0} c_n \frac{|K^{f,s}| - |K|}{s^{\frac{2}{n-1}}} &= \int_{S^{n-1}} \frac{f_K(u)^{\frac{n-2}{n-1}}}{f(N_K^{-1}(u))^{\frac{2}{n-1}}} \, d\sigma(u) \\
&= \int_{S^{n-1}} \prod_{i=1}^{n} \left[\phi_i\left(\frac{1}{f_{K_i}(u) h_{K_i}^{n+1}(u)} \right) h_{K_i}(u) f_{K_i}(u) \right]^{\frac{1}{n}} d\sigma(u) \\
&= as(\phi_1, K_1; \cdots; \phi_n, K_n),
\end{aligned}$$

where the second equality is by (2.2) and the last equality is by (1.1).

Remark. Replacing $\phi_i \in Conc(0,\infty)$ by $\psi_i \in Conv(0,\infty)$, one gets the geometric interpretation of the general mixed L_ψ-affine surface area. That is,

$$as(\psi_1, K_1; \cdots ; \psi_n, K_n) = \lim_{s\to 0} c_n \ \frac{|K^{\tilde{f},s}| - |K|}{s^{\frac{2}{n-1}}},$$

where the function $\tilde{f}$ takes the form

$$\tilde{f}(N_K^{-1}(u)) = f_K(u)^{\frac{n-2}{2}} \prod_{i=1}^{n} \left[\psi_i\left(\frac{1}{f_{K_i}(u) h_{K_i}^{n+1}(u)}\right) h_{K_i}(u) f_{K_i}(u)\right]^{\frac{1-n}{2n}}.$$

A function $\phi \in Conc(0,\infty)$ is homogeneous of degree r if $\phi(\lambda t) = \lambda^r \phi(t)$ for all $\lambda > 0$ and $t > 0$. This further implies that $\phi(t) = \phi(1)t^r$ with $r \in [0,1)$ and $\phi(t)\phi(1/t) = \phi(1)^2$. Similarly, a function $\psi \in Conv(0,\infty)$ is homogeneous of degree r if $\psi(\lambda t) = \lambda^r \psi(t)$ for all $\lambda, t > 0$. This further implies that $\psi(t) = \psi(1)t^r$ with $r \in (-\infty, 0]$ and $\psi(t)\psi(1/t) = \psi(1)^2$.

The following theorem gives the affine invariant property for the general mixed L_ϕ-affine surface area. This result was proved in [24] for all $\phi_i = \phi$ and $K_i = K$.

Theorem 2.2 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transform. For all $\phi_i \in Conc(0,\infty)$ and $K_i \in C^2_+$,*

$$as(\phi_1, TK_1; \cdots ; \phi_n, TK_n) = as(\phi_1, K_1; \cdots ; \phi_n, K_n), \ \ for \quad |det(T)| = 1.$$

If in addition, $\phi_i \in Conc(0,\infty)$ are homogeneous of degrees $r_i \in [0,1)$ for $i = 1, \cdots, n$, that is, $\phi_i(t) = \phi_i(1)t^{r_i}$, then

$$as(\phi_1, TK_1; \cdots ; \phi_n, TK_n) = |det(T)|^{1-\frac{2\sum_{i=1}^n r_i}{n}} as(\phi_1, K_1; \cdots ; \phi_n, K_n).$$

Remark. Replacing $\phi_i \in Conc(0,\infty)$ by $\psi_i \in Conv(0,\infty)$, one has the affine invariant property for the general mixed L_ψ-affine surface area; namely,

$$as(\psi_1, TK_1; \cdots ; \psi_n, TK_n) = as(\psi_1, K_1; \cdots ; \psi_n, K_n), \quad for \quad |det(T)| = 1.$$

If in addition, $\psi_i(t) = \psi_i(1)t^{r_i}$, then

$$as(\psi_1, TK_1; \cdots ; \psi_n, TK_n) = |det(T)|^{1-\frac{2\sum_{i=1}^n r_i}{n}} as(\psi_1, K_1; \cdots ; \psi_n, K_n).$$

Proof. Lemma 12 of [46] implies that, for all $u \in S^{n-1}$,

$$f_K(u) = \frac{f_{TK}(v)}{det^2(T) \ \|T^{-1t}(u)\|^{n+1}}, \tag{2.3}$$

where $v=\frac{T^{-1t}(u)}{\|T^{-1t}(u)\|}\in S^{n-1}$ with $\|\cdot\|$ standing for the Euclidean norm, A^t denotes the usual adjoint of A, and A^{-1} is the inverse of A for an operator A. On the other hand,

$$h_K(u)=\|T^{-1t}(u)\|\ h_{TK}(v). \tag{2.4}$$

Thus, we have

$$f_{TK}(v)h_{TK}^{n+1}(v)=det^2(T)\ f_K(u)h_K^{n+1}(u), \tag{2.5}$$
$$f_{TK}(v)h_{TK}(v)=det^2(T)\|T^{-1t}(u)\|^n f_K(u)h_K(u). \tag{2.6}$$

Lemma 10 in [46] implies that, for any integrable function $g:S^{n-1}\to\mathbb{R}$,

$$\int_{S^{n-1}} g(v)\,d\sigma(v)=\frac{1}{|det(T)|}\int_{S^{n-1}} g\left(\frac{T^{-1t}(u)}{\|T^{-1t}(u)\|}\right)\|T^{-1t}(u)\|^{-n}\,d\sigma(u).$$

Hence, by (2.5) and (2.6), one has

$$\begin{aligned} as(\phi_1,TK_1;\cdots;\phi_n,TK_n) &= \int_{S^{n-1}}\prod_{i=1}^n\left[\phi_i\left(\frac{1}{f_{TK_i}(v)h_{TK_i}^{n+1}(v)}\right)h_{TK_i}(v)f_{TK_i}(v)\right]^{\frac{1}{n}}d\sigma(v)\\ &= |det(T)|\int_{S^{n-1}}\prod_{i=1}^n\left[\phi_i\left(\frac{1}{det(T)^2 f_{K_i}(u)h_{K_i}^{n+1}(u)}\right)h_{K_i}(u)f_{K_i}(u)\right]^{\frac{1}{n}}d\sigma(u). \end{aligned}$$

Clearly, $|det(T)|=1$ implies $as(\phi_1,TK_1;\cdots;\phi_n,TK_n)=as(\phi_1,K_1;\cdots;\phi_n,K_n)$.

If in addition, ϕ_i are homogeneous of degrees $r_i\in[0,1)$ for $i=1,\cdots,n$, then

$$\phi_i\left(\frac{1}{det(T)^2 f_{K_i}(u)h_{K_i}^{n+1}(u)}\right)=det(T)^{-2r_i}\phi_i\left(\frac{1}{f_{K_i}(u)h_{K_i}^{n+1}(u)}\right),\ \forall u\in S^{n-1},$$

and therefore,

$$as(\phi_1,TK_1;\cdots;\phi_n,TK_n)=|det(T)|^{1-\frac{2\sum_{i=1}^n r_i}{n}}as(\phi_1,K_1;\cdots;\phi_n,K_n).$$

2.2 General mixed L_ϕ^*- and L_ψ^*-affine surface areas

In [24], Ludwig showed that the L_p affine surface area for $p<-n$ is associated with the L_ψ^*-affine surface area for $\psi\in Conv(0,\infty)$. Here, we define the general mixed L_ψ^*-affine surface area by

$$as^*(\psi_1,K_1;\cdots;\psi_n,K_n)=\int_{S^{n-1}}\prod_{i=1}^n\left[\frac{\psi_i(f_{K_i}(u)h_{K_i}^{n+1}(u))}{h_{K_i}(u)^n}\right]^{\frac{1}{n}}d\sigma(u),$$

for $\psi_i \in Conv(0,\infty)$ and $K_i \in C^2_+$, $i=1,\cdots,n$. As above, $as^*_\psi(K_1,\cdots,K_n) = as^*(\psi,K_1;\cdots;\psi,K_n)$ and $as^*_\psi(K) = as^*_\psi(K,\cdots,K)$. If all K_i coincide with K, we use $as^*(\psi_1,\cdots,\psi_n;K)$ instead of $as^*(\psi_1,K;\cdots;\psi_n,K)$. If $\psi(t)=t^{\frac{n}{n+p}}$ for $p<-n$, then $as^*_\psi(K_1,\cdots,K_n)=as_p(K_1,\cdots,K_n)$ and hence, $as^*_\psi(K)$ is the L_p affine surface area for $p<-n$. In particular, $as^*(\psi_1,\cdots,\psi_n;B^n_2)=\big[\psi_1(1)\cdots\psi_n(1)\big]^{\frac{1}{n}} n|B^n_2|$.

The following proposition gives the duality relation between L_ψ- and L^*_ψ-affine surface areas.

Proposition 2.1 *Let all $K_i \in C^2_+$ be convex bodies, such that, $K_i=\lambda_i K$ for some convex body $K\in C^2_+$ and $\lambda_i>0$, $i=1,\cdots,n$. For all $\psi_i \in Conv(0,\infty)$,*

$$as^*(\psi_1,K_1;\cdots;\psi_n,K_n) = as(\psi_1,K_1^\circ;\cdots;\psi_n,K_n^\circ).$$

In particular, $as^*(\psi_1,\cdots,\psi_n;K) = as(\psi_1,\cdots,\psi_n;K^\circ)$.

Proof. Define $y: S^{n-1}\to\partial K^\circ$ by $y(u)=\rho_{K^\circ}(u)u$ with $\rho_{K^\circ}(u)$ the radius function of K° at the direction u, that is, $\rho_{K^\circ}(u)=\max\{\lambda>0:\lambda u\in K^\circ\}$. Note that $h_K(u)\rho_{K^\circ}(u)=1$ for all directions $u\in S^{n-1}$. The Jacobian Jy is (see e.g. [20])

$$Jy(u) = \frac{\rho_{K^\circ}(u)^{n-1}}{\langle u, N_{K^\circ}(\rho_{K^\circ}(u)u)\rangle}, \quad a.s. \ \ on \ \ S^{n-1}.$$

The area formula (see e.g. [12]) implies that for every a.s. defined function $g: S^{n-1}\to[0,\infty]$, one has $\int_{S^{n-1}} g(u)Jy(u)\,d\sigma(u)=\int_{\partial K^\circ} g\big(\frac{y}{\|y\|}\big)\,d\mu_{K^\circ(y)}$. Setting

$$g(u) = \prod_{i=1}^n \left[\psi_i\left(\frac{1}{\lambda_i^{2n} f_K(u) h_K^{n+1}(u)}\right) h_K(u) f_K(u)\right]^{\frac{1}{n}} \frac{\langle u, N_{K^\circ}(y(u))\rangle}{\rho_{K^\circ}(u)^{n-1}},$$

one has

$$\begin{aligned}
&as(\psi_1,K_1;\cdots;\psi_n,K_n) = as(\psi_1,\lambda_1 K;\cdots;\psi_n,\lambda_n K)\\
&= \lambda_1\cdots\lambda_n \int_{S^{n-1}} \prod_{i=1}^n \left[\psi_i\left(\frac{1}{\lambda_i^{2n} f_K(u) h_K^{n+1}(u)}\right) h_K(u) f_K(u)\right]^{\frac{1}{n}} d\sigma(u)\\
&= \lambda_1\cdots\lambda_n \int_{K^\circ} \prod_{i=1}^n \left[\psi_i\left(\frac{\lambda_i^{-2n}}{f_K\big(\frac{y}{\|y\|}\big) h_K^{n+1}\big(\frac{y}{\|y\|}\big)}\right) h_K\Big(\frac{y}{\|y\|}\Big) f_K\Big(\frac{y}{\|y\|}\Big)\right]^{\frac{1}{n}} \frac{\langle y, N_{K^\circ}(y)\rangle}{\rho_{K^\circ}(\frac{y}{\|y\|})^n}\, d\mu_{K^\circ}(y)\\
&= \lambda_1\cdots\lambda_n \int_{K^\circ} \prod_{i=1}^n \left[\psi_i\left(\frac{\lambda_i^{-2n}}{f_K\big(\frac{y}{\|y\|}\big) h_K^{n+1}\big(\frac{y}{\|y\|}\big)}\right)\right]^{\frac{1}{n}} h_K^{n+1}\Big(\frac{y}{\|y\|}\Big) f_K\Big(\frac{y}{\|y\|}\Big) \langle y, N_{K^\circ}(y)\rangle\, d\mu_{K^\circ}(y),
\end{aligned} \tag{2.7}$$

where the third equality is by $\|y\| = \rho_{K^\circ}(\frac{y}{\|y\|})$ and the last equality is by $h_K(u) = \rho_{K^\circ}(u)$ for all directions $u \in S^{n-1}$.

Now we let $v = N_{K^\circ}(y)$. Hug [20] proved that for almost all $y \in \partial K^\circ$,

$$h_K^{n+1}\left(\frac{y}{\|y\|}\right) f_K\left(\frac{y}{\|y\|}\right) = \frac{1}{h_{K^\circ}^{n+1}(v) f_{K^\circ}(v)}.$$

Combining with $d\mu_{K^\circ}(y) = f_{K^\circ}(v)\, d\sigma(v)$ and by $(\lambda_i K)^\circ = \lambda_i^{-1} K^\circ$, the equality (2.7) equals to

$$\lambda_1 \cdots \lambda_n \int_{S^{n-1}} \prod_{i=1}^n \left[\psi_i\left(\lambda_i^{-2n} h_{K^\circ}^{n+1}(v) f_{K^\circ}(v)\right)\right]^{\frac{1}{n}} \frac{d\sigma(v)}{h_{K^\circ}^n(v)} = as^*(\psi_1, K_1^\circ; \cdots; \psi_n, K_n^\circ),$$

which completes the proof.

Remark. When all $\psi_i = \psi$, this result was proved in [24], i.e., $as_\psi^*(K) = as_\psi(K^\circ)$ for all $K \in C_+^2$ and $\psi \in Conv(0,\infty)$. In particular, if $\psi(t) = t^{\frac{n}{n+p}}$ for $p < -n$, then $as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$ [52]. In general, one *cannot* expect $as^*(\psi_1, K_1; \cdots; \psi_n, K_n) = as(\psi_1, K_1^\circ; \cdots; \psi_n, K_n^\circ)$ even if all ψ_i are equal to some $\psi \in Conv(0,\infty)$ and all K_i are ellipsoids. To this end, let $n = 2$ and $\psi(t) = \frac{1}{t}$. For any 2-dimensional convex body $K \in C_+^2$,

$$as_\psi(K, B_2^2) = \int_{S^1} \left[\psi\left(\frac{1}{f_K(u) h_K^3(u)}\right) h_K(u) f_K(u)\right]^{\frac{1}{2}} d\sigma_1(u) = \int_{S^1} h_K^2(u) f_K(u)\, d\sigma_1(u),$$

where σ_1 refers to the spherical measure of S^1. On the other hand,

$$as_\psi^*(K^\circ, B_2^2) = \int_{S^1} \frac{d\sigma_1(u)}{\sqrt{f_{K^\circ}(u) h_{K^\circ}^5(u)}}.$$

Now let the (invertible) affine map $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $T(x_1, x_2) = (x_1, 2x_2)$. Then, by formulas (2.3) and (2.4),

$$h_{TB_2^2}(u) = \|Tu\|, \quad f_{TB_2^2}(u) = \frac{4}{\|Tu\|^3}, \quad \forall u \in S^1.$$

Therefore, one has

$$as_\psi(TB_2^2, B_2^2) = \int_{S^1} f_{TB_2^2}(u) h_{TB_2^2}^2(u)\, d\sigma_1(u) = \int_{S^1} \frac{4}{\|Tu\|}\, d\sigma_1(u) = \int_{S^1} \frac{4}{\sqrt{1+3u_2^2}}\, d\sigma_1(u).$$

As $(TB_2^2)^\circ = T^{-1t} B_2^2$, one has

$$h_{(TB_2^2)^\circ}(u) = \|T^{-1}u\|, \quad f_{(TB_2^2)^\circ}(u) = \frac{4}{\|T^{-1}u\|^3}, \quad \forall u \in S^1,$$

and hence

$$as^*_\psi((TB_2^2)^\circ, B_2^2) = \int_{S^1} \frac{d\sigma_1(u)}{\sqrt{4\|T^{-1}u\|^2}} = \int_{S^1} \frac{1}{\sqrt{1+3u_1^2}}\, d\sigma_1(u).$$

Clearly $as_\psi(TB_2^2, B_2^2) = 4as^*_\psi((TB_2^2)^\circ, B_2^2)$ by the rotational invariance of the spherical measure σ_1.

For all $\phi_i \in Conc(0,\infty)$ and all $K_i \in C^2_+$, we define the general mixed L^*_ϕ-affine surface area by

$$as^*(\phi_1, K_1; \cdots; \phi_n, K_n) = \int_{S^{n-1}} \prod_{i=1}^n \left[\frac{\phi_i(f_{K_i}(u)h^{n+1}_{K_i}(u))}{h_{K_i}(u)^n}\right]^{\frac{1}{n}} d\sigma(u).$$

Similarly, let $as^*_\phi(K_1, \cdots, K_n) = as^*(\phi, K_1; \cdots; \phi, K_n)$, $as^*_\phi(K) = as^*_\phi(K, \cdots, K)$, and $as^*(\phi_1, \cdots, \phi_n; K) = as^*(\phi_1, K; \cdots; \phi_n, K)$. In particular, $as^*(\phi_1, \cdots, \phi_n; B_2^n) = \big[\phi_1(1)\cdots\phi_n(1)\big]^{\frac{1}{n}} n|B_2^n|$.

Proposition 2.2 *Let $K_i \in C^2_+$ be convex bodies, such that, $K_i = \lambda_i K$ for some convex body $K \in C^2_+$ and $\lambda_i > 0$, $i = 1, \cdots, n$. For all $\phi_i \in Conc(0,\infty)$,*

$$as^*(\phi_1, K_1; \cdots; \phi_n, K_n) = as(\phi_1, K_1^\circ; \cdots; \phi_n, K_n^\circ).$$

In particular, $as^*(\phi_1, \cdots, \phi_n; K) = as(\phi_1, \cdots, \phi_n; K^\circ)$.

Remark. The proof of this proposition is similar to that of Proposition 2.1. An immediate consequence is $as^*_\phi(K) = as_\phi(K^\circ)$ for $K \in C^2_+$ and $\phi \in Conc(0,\infty)$. In particular, if $\phi(t) = t^{\frac{p}{n+p}}$ for $p \geq 0$, one obtains the duality formula $as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$ [20, 52]. Hence the L^*_ϕ-affine surface area can be viewed as a generalization of the L_p affine surface area for $p \geq 0$. As above, one *cannot* expect, in general, $as^*(\phi_1, K_1; \cdots; \phi_n, K_n) = as(\phi_1, K_1^\circ; \cdots; \phi_n, K_n^\circ)$ even if all $\phi_i = \phi$ for some $\phi \in Conc(0,\infty)$ and all K_i are ellipsoids.

The following theorem gives a geometric interpretation for the general mixed L^*_ϕ-affine surface area. Similar geometric interpretations can also be obtained by the surface body [46, 52].

Theorem 2.3 *Let $K, K_i \in C^2_+$ and $\psi_i \in Conv(0,\infty)$, $i = 1, \cdots, n$. Let $g : \partial K \to \mathbb{R}$ be the function*

$$g(N_K^{-1}(u)) = f_K(u)^{\frac{n-2}{2}} \prod_{i=1}^n \left[\frac{\psi_i(f_{K_i}(u)h^{n+1}_{K_i}(u))}{h_{K_i}(u)^n}\right]^{\frac{1-n}{2n}}.$$

Let $c_n = 2|B_2^{n-1}|^{\frac{2}{n-1}}$*. Then,*

$$as^*(\psi_1, K_1; \cdots; \psi_n, K_n) = \lim_{s\to 0} c_n \ \frac{|K^{g,s}| - |K|}{s^{\frac{2}{n-1}}}.$$

Remark. Replacing $\psi_i \in Conv(0,\infty)$ by $\phi_i \in Conc(0,\infty)$, one gets the geometric interpretation for the general mixed L_ϕ^*-affine surface area. That is,

$$as^*(\phi_1, K_1; \cdots; \phi_n, K_n) = \lim_{s\to 0} c_n \ \frac{|K^{\tilde{g},s}| - |K|}{s^{\frac{2}{n-1}}},$$

where the function $\tilde{g}$ takes the form

$$\tilde{g}(N_K^{-1}(u)) = f_K(u)^{\frac{n-2}{2}} \prod_{i=1}^{n} \left[\frac{\phi_i(f_{K_i}(u)h_{K_i}^{n+1}(u))}{h_{K_i}(u)^n}\right]^{\frac{1-n}{2n}}.$$

The general mixed L_ψ^*-affine surface area is also affine invariant. For all $\psi_i = \psi$ and $K_i = K$, this result was proved in [24]. The proof is similar to that of Theorem 2.2, and we omit it.

Theorem 2.4 *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be an invertible linear transform. For all* $\psi_i \in Conv(0,\infty)$ *and* $K_i \in C_+^2$*, one has*

$$as^*(\psi_1, TK_1; \cdots; \psi_n, TK_n) = as^*(\psi_1, K_1; \cdots; \psi_n, K_n), \ \ for \quad |det(T)| = 1.$$

If in addition, all $\psi_i \in Conv(0,\infty)$ *are homogeneous of degrees* $r_i \in (-\infty, 0]$*, then*

$$as^*(\psi_1, TK_1; \cdots; \psi_n, TK_n) = |det(T)|^{\frac{2\sum_{i=1}^n r_i}{n} - 1} as^*(\psi_1, K_1; \cdots; \psi_n, K_n).$$

Remark. Similar results hold for the general mixed L_ϕ^*-affine surface area, i.e.,

$$as^*(\phi_1, TK_1; \cdots; \phi_n, TK_n) = as^*(\phi_1, K_1; \cdots; \phi_n, K_n), \ \ for \quad |det(T)| = 1.$$

If in addition, $\phi_i \in Conc(0,\infty)$ are homogeneous of degrees $r_i \in [0, 1)$, then

$$as^*(\phi_1, TK_1; \cdots; \phi_n, TK_n) = |det(T)|^{\frac{2\sum_{i=1}^n r_i}{n} - 1} as^*(\phi_1, K_1; \cdots; \phi_n, K_n).$$

3 Inequalities

A general version of the classical Alexandrov-Fenchel inequalities for mixed volumes (see [1, 8, 44]) can be written as

$$\prod_{i=0}^{m-1} V(K_1,\cdots,K_{n-m},\underbrace{K_{n-i},\cdots,K_{n-i}}_{m}) \le V^m(K_1,\cdots,K_n).$$

Here the analogous inequalities for general mixed affine surface areas are proved. We refer readers to the references [28, 29, 31, 53] for similar results related to the mixed p-affine surfaces area.

Proposition 3.1 *Let all $K_i \in C_+^2$ and $\phi_i \in Conc(0,\infty)$. Then, for $1 \le m \le n$,*

$$as^m(\phi_1,K_1;\cdots;\phi_n,K_n) \le \prod_{i=0}^{m-1} as(\phi_1,K_1;\cdots;\phi_{n-m},K_{n-m};\underbrace{\phi_{n-i},K_{n-i};\cdots;\phi_{n-i},K_{n-i}}_{m}).$$

Equality holds if (1) all K_i coincide and $\phi_i = \lambda_i\phi_n$ for some $\lambda_i > 0$, $i = n-m+1,\cdots,n$, or (2) $\phi_i = \lambda_i\phi_n$, $K_i = \eta_i K_n$ for some $\lambda_i,\eta_i > 0$, $i = n-m+1,\cdots,n$, and ϕ_n is homogeneous of degree $r \in [0,1)$. The equality holds trivially if $m = 1$.

In particular, if $m = n$,

$$as^n(\phi_1,K_1;\cdots;\phi_n,K_n) \le as_{\phi_1}(K_1)\cdots as_{\phi_n}(K_n). \tag{3.8}$$

Proof. Let us put

$$g_0(u) = \prod_{i=1}^{n-m}\left[\phi_i\left(\frac{1}{f_{K_i}(u)h_{K_i}^{n+1}(u)}\right)h_{K_i}(u)f_{K_i}(u)\right]^{\frac{1}{n}},$$

and for $j = 0,\cdots,m-1$, put

$$g_{j+1}(u) = \left[\phi_{n-j}\left(\frac{1}{f_{K_{n-j}}(u)h_{K_{n-j}}^{n+1}(u)}\right)h_{K_{n-j}}(u)f_{K_{n-j}}(u)\right]^{\frac{1}{n}}.$$

By Hölder's inequality (see [18])

$$\begin{aligned} as(\phi_1,K_1;\cdots;\phi_n,K_n) &= \int_{S^{n-1}} g_0(u)g_1(u)\cdots g_m(u)\,d\sigma(u) \\ &\le \prod_{j=0}^{m-1}\left(\int_{S^{n-1}} g_0(u)g_{j+1}^m(u)\,d\sigma(u)\right)^{\frac{1}{m}} \\ &= \prod_{j=0}^{m-1} as^{\frac{1}{m}}(\phi_1,K_1;\cdots;\phi_{n-m},K_{n-m};\underbrace{\phi_{n-j},K_{n-j};\cdots;\phi_{n-j},K_{n-j}}_{m}). \end{aligned}$$

As $K_i \in C_+^2$ and $\phi_i \neq 0$, $g_k(u) > 0$ for all $k = 0, 1, \cdots, m$ and all $u \in S^{n-1}$. Therefore, equality in Hölder's inequality holds if and only if $g_0(u) g_{j+1}^m(u) = \lambda_{n-j}^m g_0(u) g_1^m(u)$ for some $\lambda_{n-j} > 0$ and all $0 \leq j \leq m-1$. This condition holds true if (1) all K_i coincide and $\phi_i = \lambda_i \phi_n$ for some $\lambda_i > 0$, $i = n-m+1, \cdots, n$, or (2) $\phi_i = \lambda_i \phi_n$, $K_i = \eta_i K_n$ for some $\lambda_i, \eta_i > 0$, $i = n-m+1, \cdots, n$, and ϕ_n is homogeneous of degree $r \in [0, 1)$.

Remark. When all $\phi_i(t) = t^{\frac{p}{n+p}}$ for $p \geq 0$, this recovers the Alexandrov-Fenchel inequalities for mixed p-affine surface areas with $p \geq 0$ [53]. When all K_i coincide with K, then

$$as^m(\phi_1, \cdots, \phi_n; K) \leq \prod_{i=0}^{m-1} as(\phi_1, \cdots, \phi_{n-m}, \underbrace{\phi_{n-i}, \cdots, \phi_{n-i}}_{m}; K),$$

and equality holds true if $\phi_i = \lambda_i \phi_n$ for some $\lambda_i > 0$, $i = n-m+1, \cdots, n$. When all ϕ_i coincide with ϕ, then

$$as_\phi^m(K_1, \cdots, K_n) \leq \prod_{i=0}^{m-1} as_\phi(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}),$$

and equality holds true if (1) all K_i, $i = n-m+1, \cdots, n$ coincide, or (2) $K_i = \eta_i K_n$ for some $\eta_i > 0$, $i = n-m+1, \cdots, n$ and ϕ is homogeneous of degree $r \in [0, 1)$.

Remark. With slight modification, one can get analogous Alexandrov-Fenchel inequality for the general mixed L_ϕ^*-affine surface area, namely,

$$\big[as^*(\phi_1, K_1; \cdots; \phi_n, K_n)\big]^m \leq \prod_{i=0}^{m-1} as^*(\phi_1, K_1; \cdots; \phi_{n-m}, K_{n-m}; \underbrace{\phi_{n-i}, K_{n-i}; \cdots; \phi_{n-i}, K_{n-i}}_{m}).$$

In particular, if $m = n$,

$$\big[as^*(\phi_1, K_1; \cdots; \phi_n, K_n)\big]^n \leq as^*_{\phi_1}(K_1) \cdots as^*_{\phi_n}(K_n). \tag{3.9}$$

Replacing $\phi_i \in Conc(0, \infty)$ by $\psi_i \in Conv(0, \infty)$, one gets the Alexandrov-Fenchel inequalities for the general mixed L_ψ- and L_ψ^*-affine surface areas.

Blaschke-Santaló inequality states that, for all convex body K with centroid at the origin, $|K||K^\circ| \leq |B_2^n|^2$; equality holds if and only if K is an ellipsoid. This fundamental inequality has been generalized to the L_p affine surface area and mixed p-affine surface area [31, 52, 53]. Here, we prove the Santaló-type inequalities for the general mixed L_ϕ-affine surface area. The Santaló-type inequality for the general mixed L_ϕ^*-affine surface area can be achieved in the same way.

Theorem 3.1 *Let $K_i \in C^2_+$ and $\phi_i \in Conc(0,\infty)$ be homogeneous of degrees $r_i \in [0,1)$. Then*

$$as(\phi_1, K_1; \cdots; \phi_n, K_n) as(\phi_1, K_1^\circ; \cdots; \phi_n, K_n^\circ) \le n^2 \prod_{i=1}^n \left[\phi_i(1)^2 \ |K_i| \ |K_i^\circ|\right]^{\frac{1}{n}}.$$

Moreover, if all $K_i \in C^2_+$ have centroid at the origin, then

$$as(\phi_1, K_1; \cdots; \phi_n, K_n) as(\phi_1, K_1^\circ; \cdots; \phi_n, K_n^\circ) \le \left[as(\phi_1, B_2^n; \cdots; \phi_n, B_2^n)\right]^2,$$

with equality if all K_i are origin-symmetric ellipsoids that are dilates of one another.

Proof. The following inequality was proved in [24] and it actually holds true for all $K \in C^2_+$: if $\phi \in Conc(0,\infty)$, then

$$as_\phi(K) \le n|K|\phi\left(\frac{|K^\circ|}{|K|}\right). \tag{3.10}$$

Thus, for any $K \in C^2_+$,

$$as_\phi(K) as_\phi(K^\circ) \le n^2 |K||K^\circ| \phi\left(\frac{|K^\circ|}{|K|}\right)\phi\left(\frac{|K|}{|K^\circ|}\right).$$

Combining with inequality (3.8), one has, for all $K_i \in C^2_+$,

$$\begin{aligned} as(\phi_1,K_1;\cdots;\phi_n,K_n) as(\phi_1,K_1^\circ;\cdots;\phi_n,K_n^\circ) &\le \prod_{i=1}^n \left[as_{\phi_i}(K_i) as_{\phi_i}(K_i^\circ)\right]^{\frac{1}{n}} \\ \le n^2 \prod_{i=1}^n \left[|K_i||K_i^\circ| \phi_i\left(\frac{|K_i^\circ|}{|K_i|}\right)\phi_i\left(\frac{|K_i|}{|K_i^\circ|}\right)\right]^{\frac{1}{n}} &= n^2 \prod_{i=1}^n \left[\phi_i(1)^2 \ |K_i| \ |K_i^\circ|\right]^{\frac{1}{n}}. \end{aligned} \tag{3.11}$$

Here the equality follows from all ϕ_i being of homogenous degrees $r_i \in [0,1)$.

If all $K_i \in C^2_+$ have centroid at the origin, one can employ Blaschke-Santaló inequality to inequality (3.11) and get

$$as(\phi_1,K_1;\cdots;\phi_n,K_n) as(\phi_1,K_1^\circ;\cdots;\phi_n,K_n^\circ) \le n^2 \prod_{i=1}^n \left[\phi_i(1)|B_2^n|\right]^{\frac{2}{n}} = \left[as(\phi_1,\cdots,\phi_n; B_2^n)\right]^2.$$

Clearly, equality holds true if K_i are origin-symmetric ellipsoids that are dilates of one another.

Let $K \in C^2_+$ be a convex body with centroid at the origin and B_K be the origin-symmetric (Euclidean) ball such that $|B_K| = |K|$. For the L_ϕ-affine surface

area with $\phi \in Conc(0,\infty)$, Ludwig proved the affine isoperimetric inequality [24]; namely, $as_\phi(K) \le as_\phi(B_K)$ with equality if and only if K is an ellipsoid. If we further assume that ϕ is homogeneous of degree $r \in [0,1)$, then the affine isoperimetric inequality for L_ϕ-affine surface area may be stated as

$$\left(\frac{as_\phi(K)}{as_\phi(B_2^n)}\right) \le \left(\frac{|K|}{|B_2^n|}\right)^{1-2r}, \tag{3.12}$$

with equality if and only if K is an ellipsoid. In fact, let $\lambda = \frac{|B_2^n|}{|K|}$ and $\tilde{K} = \lambda^{\frac{1}{n}} K$, then $|\tilde{K}| = |B_2^n|$. Employing the affine isoperimetric inequality in [24] to $\tilde{K}$, one has $\frac{as_\phi(\tilde{K})}{as_\phi(B_2^n)} \le 1$. By Theorem 2.2, one has

$$\frac{as_\phi(\tilde{K})}{as_\phi(B_2^n)} = \frac{as_\phi(\lambda^{\frac{1}{n}} K)}{as_\phi(B_2^n)} = \lambda^{1-2r}\left(\frac{as_\phi(K)}{as_\phi(B_2^n)}\right) \le 1,$$

which is equivalent to the formula (3.12). There is an equality if and only if K is an ellipsoid. When $\phi(t) = t^{\frac{p}{n+p}}$ for $p > 0$, one gets the L_p affine isoperimetric inequality for $p > 0$ [31, 52]. For $p = 0$, one has equality instead of inequality.

Next, we prove the affine isoperimetric inequalities for general mixed affine surface areas. Hereafter, we always let B_{K_i} be the origin-symmetric (Euclidean) ball s.t. $|B_{K_i}| = |K_i|$ for all i.

Theorem 3.2 *Let all $K_i \in C_+^2$ be convex bodies with centroid at the origin.*

(i): If all $\phi_i \in Conc(0,\infty)$, then

$$as(\phi_1, K_1; \cdots; \phi_n, K_n) \le as(\phi_1, B_{K_1}; \cdots; \phi_n, B_{K_n}).$$

Equality holds if all K_i are ellipsoids that are dilates of one another.

(ii) For all $\phi_i \in Conc(0,\infty)$ with homogeneous degrees $r_i \in [0,1)$,

$$\left(\frac{as(\phi_1, K_1; \cdots; \phi_n, K_n)}{as(\phi_1, \cdots, \phi_n; B_2^n)}\right)^n \le \prod_{i=1}^n \left(\frac{|K_i|}{|B_2^n|}\right)^{1-2r_i},$$

with equality if all K_i are ellipsoids that are dilates of one another.

Proof.

(i). Let all $K_i \in C_+^2$ be convex bodies with centroid at the origin and $|K_i| = |B_{K_i}|$ for all i. It is easy to verify that $as_{\phi_1}(B_{K_1}) \cdots as_{\phi_n}(B_{K_n}) = \left[as(\phi_1, B_{K_1}; \cdots; \phi_n, B_{K_n})\right]^n$.

By inequality (3.8) and the affine isoperimetric inequality for L_ϕ-affine surface area, one has, for all $\phi_i \in Conc(0,\infty)$,

$$\begin{aligned} as^n(\phi_1,K_1;\cdots;\phi_n,K_n) &\leq as_{\phi_1}(K_1)\cdots as_{\phi_n}(K_n) \leq as_{\phi_1}(B_{K_1})\cdots as_{\phi_n}(B_{K_n}) \\ &= \big[as(\phi_1,B_{K_1};\cdots;\phi_n,B_{K_n})\big]^n. \end{aligned}$$

By the affine invariant property in Theorem 2.2, the equality holds true if K_i are ellipsoids that are dilates of one another.

(ii). Recall that $\big[as(\phi_1,\cdots,\phi_n;B_2^n)\big]^n = as_{\phi_1}(B_2^n)\cdots as_{\phi_n}(B_2^n)$. By inequalities (3.8) and (3.12), one has

$$\left(\frac{as(\phi_1,K_1;\cdots;\phi_n,K_n)}{as(\phi_1,\cdots,\phi_n;B_2^n)}\right)^n \leq \frac{as_{\phi_1}(K_1)\cdots as_{\phi_n}(K_n)}{as_{\phi_1}(B_2^n)\cdots as_{\phi_n}(B_2^n)} \leq \prod_{i=1}^n \left(\frac{|K_i|}{|B_2^n|}\right)^{1-2r_i}.$$

If all K_i are ellipsoids that are dilates of one another, the equality holds true.

Remark 3.1 *If all $\phi_i(t) = t^{\frac{p}{n+p}}$ for $p \geq 0$, one recovers affine isoperimetric inequalities for mixed p-affine surface areas [53]. One cannot expect to get strictly positive lower bounds in Theorem 3.2. Let the convex body $K(R,\varepsilon) \subset \mathbb{R}^2$ be the intersection of four Euclidean balls with radius R centered at $(\pm(R-1),0)$, $(0,\pm(R-1))$, R arbitrarily large. We then "round" the corners by putting there arcs of Euclidean balls of arbitrarily small radius ε, and "bridge" between the R-arcs and ε-arcs by C^2_+-arcs on a set of arbitrarily small measure to obtain a convex body in C^2_+ [52]. Then $as_\phi(K(R,\varepsilon)) \leq 16R^{-\frac{p}{p+2}} + 4\pi\varepsilon^{\frac{2}{2+p}}$ for $\phi(t) = t^{\frac{p}{n+p}}$ with $p > 0$, which goes to 0 as $R \to \infty$ and $\varepsilon \to 0$. Choose now R_i and ε_i, $1 \leq i \leq n$, such that $R_i \to \infty$ and $\varepsilon_i \to 0$, and let $K_i = K(R_i,\varepsilon_i)$ for $i = 1,\cdots,n$. Let $\phi_i(t) = t^{\frac{p_i}{n+p_i}}$ for $p_i > 0$, by inequality (3.8), $as^n(\phi_1,K_1;\cdots;\phi_n,K_n) \leq \prod_{i=1}^n as_{\phi_i}(K_i)$, and thus $as(\phi_1,K_1;\cdots;\phi_n,K_n) \to 0$.*

We can prove the following affine isoperimetric inequality for the general mixed L_ϕ^*-affine surface area.

Theorem 3.3 *Let all $K_i \in C^2_+$ be convex bodies with centroid at the origin.*

(i): If all $\phi_i \in Conc(0,\infty)$, then

$$as^*(\phi_1,K_1;\cdots;\phi_n,K_n) \leq as^*(\phi_1,(B_{K_1^\circ})^\circ;\cdots;\phi_n,(B_{K_n^\circ})^\circ).$$

Equality holds if K_i are ellipsoids that are dilates of one another.

(ii) For all $\phi_i \in Conc(0,\infty)$ with homogeneous degrees $r_i \in [0,1)$,

$$\left(\frac{as^*(\phi_1, K_1; \cdots; \phi_n, K_n)}{as^*(\phi_1, \cdots, \phi_n; B_2^n)}\right)^n \le \prod_{i=1}^n \left(\frac{|K_i|}{|B_2^n|}\right)^{2r_i-1},$$

with equality if K_i are ellipsoids that are dilates of one another.

Proof.

(i). By inequality (3.9), Proposition 2.2, and the affine isoperimetric inequality for the L_ϕ-affine surface area, one gets,

$$\begin{aligned}
[as^*(\phi_1, K_1; \cdots; \phi_n, K_n)]^n &\le as^*_{\phi_1}(K_1)\cdots as^*_{\phi_n}(K_n) = as_{\phi_1}(K_1^\circ)\cdots as_{\phi_n}(K_n^\circ)\\
&\le as_{\phi_1}(B_{K_1^\circ})\cdots as_{\phi_n}(B_{K_n^\circ}) = \big[as(\phi_1, B_{K_1^\circ}; \cdots; \phi_n, B_{K_n^\circ})\big]^n\\
&= \big[as^*(\phi_1, (B_{K_1^\circ})^\circ; \cdots; \phi_n, (B_{K_n^\circ})^\circ)\big]^n,
\end{aligned}$$

where the last equality follows Proposition 2.2. Following the affine invariant property, equality holds if K_i are ellipsoids that are dilates of one another.

(ii). Similarly, by inequalities (3.9), (3.10), and Proposition 2.2, one has

$$\begin{aligned}
\big[as^*(\phi_1, K_1; \cdots; \phi_n, K_n)\big]^n &\le as^*_{\phi_1}(K_1)\cdots as^*_{\phi_n}(K_n) = as_{\phi_1}(K_1^\circ)\cdots as_{\phi_n}(K_n^\circ)\\
&\le n^n \prod_{i=1}^n \big[|K_i^\circ|\phi_i(|K_i|/|K_i^\circ|)\big] = n^n \prod_{i=1}^n \big[\phi_i(1)|K_i^\circ|^{1-r_i}|K_i|^{r_i}\big].
\end{aligned}$$

By Blaschke-Santaló inequality, i.e., $|K^\circ||K| \le |B_2^n|^2$, and $r_i \in [0,1)$, one gets

$$\big[as^*(\phi_1, K_1; \cdots; \phi_n, K_n)\big]^n \le n^n \prod_{i=1}^n \big[\phi_i(1)|K_i|^{2r_i-1}|B_2^n|^{2-2r_i}\big].$$

Equivalently, by $as^*(\phi_1, \cdots, \phi_n; B_2^n) = \big[\phi_1(1)\cdots\phi_n(1)\big]^{\frac{1}{n}} n|B_2^n|$, one has

$$\left(\frac{as^*(\phi_1, K_1; \cdots; \phi_n, K_n)}{as^*(\phi_1, \cdots, \phi_n; B_2^n)}\right)^n \le \prod_{i=1}^n \left(\frac{|K_i|}{|B_2^n|}\right)^{2r_i-1}.$$

Clearly, equality holds true if all K_i are ellipsoids that are dilates of one another.

Remark 3.2 *One cannot expect to get strictly positive lower bounds in Theorem 3.3. Let the convex bodies $K_i = K(R_i, \varepsilon_i) \in C_+^2$ be as in Remark (3.1) with $R_i \to \infty$ and $\varepsilon_i \to 0$. Let $\phi_i(t) = t^{\frac{p_i}{n+p_i}}$ with $p_i > 0, i = 1, \cdots, n$. By inequality (3.9) and Proposition 2.1, one gets $\big[as^*(\phi_1, K_1^\circ; \cdots; \phi_n, K_n^\circ)\big]^n \le \prod_{i=1}^n as^*_{\phi_i}(K_i^\circ) = \prod_{i=1}^n as_{\phi_i}(K_i)$, which goes to 0 as $R_i \to \infty$ and $\varepsilon \to 0$.*

4 General i-th mixed affine surface areas and related inequalities

Let i be a real number and $\phi_1, \phi_2 \in Conc(0,\infty)$. The general i-th mixed L_ϕ-affine surface area of $K, L \in C_+^2$ is defined as

$$as_i(\phi_1,K;\phi_2,L)=\int_{S^{n-1}}\left[\phi_1\left(\frac{f_K^{-1}(u)}{h_K^{n+1}(u)}\right)h_K(u)f_K(u)\right]^{\frac{n-i}{n}}\left[\phi_2\left(\frac{f_L^{-1}(u)}{h_L^{n+1}(u)}\right)h_L(u)f_L(u)\right]^{\frac{i}{n}}d\sigma(u). \tag{4.13}$$

If i is an integer number with $0 \le i \le n$, then

$$as_i(\phi_1, K; \phi_2, L) = as(\underbrace{\phi_1, K; \cdots ; \phi_1, K}_{n-i}; \underbrace{\phi_2, L; \cdots ; \phi_2, L}_{i}).$$

In particular, $as_i(\phi_1, K; \phi_2, L) = as_{n-i}(\phi_2, L; \phi_1, K)$, $as_0(\phi_1, K; \phi_2, L) = as_{\phi_1}(K)$, and $as_n(\phi_1, K; \phi_2, L) = as_{\phi_2}(L)$. If $\phi_1(t) = \phi_2(t) = t^{\frac{p}{n+p}}$ for $p \ge 0$, one gets the i-th mixed p-affine surface area [29, 50, 53]. Note that the i-th mixed p-affine surface area includes the surface area of K as a special case; namely the surface area of K is $as_{-1}(\phi, K; \phi, B_2^n)$ with $\phi(t) = t^{\frac{1}{n+1}}$. Obviously, the general i-th mixed L_ϕ-affine surface area satisfies the affine invariant property as in Theorem 2.2. Similarly, we can define the general i-th mixed L_ϕ^*-affine surface area of $K, L \in C_+^2$ as

$$as_i^*(\phi_1,K;\phi_2,L)=\int_{S^{n-1}}\left[\frac{\phi_1(f_K(u)h_K^{n+1}(u))}{h_K^n(u)}\right]^{\frac{n-i}{n}}\left[\frac{\phi_2(f_L(u)h_L^{n+1}(u))}{h_L^n(u)}\right]^{\frac{i}{n}}d\sigma(u).$$

For $\psi_1, \psi_2 \in Conv(0,\infty)$, the general i-th mixed L_ψ-affine surface area of $K, L \in C_+^2$ is defined as

$$as_i(\psi_1,K;\psi_2,L)=\int_{S^{n-1}}\left[\psi_1\left(\frac{f_K^{-1}(u)}{h_K^{n+1}(u)}\right)h_K(u)f_K(u)\right]^{\frac{n-i}{n}}\left[\psi_2\left(\frac{f_L^{-1}(u)}{h_L^{n+1}(u)}\right)h_L(u)f_L(u)\right]^{\frac{i}{n}}d\sigma(u),$$

and the general i-th mixed L_ψ^*-affine surface area of $K, L \in C_+^2$ is defined as

$$as_i^*(\psi_1,K;\psi_2,L)=\int_{S^{n-1}}\left[\frac{\psi_1(f_K(u)h_K^{n+1}(u))}{h_K^n(u)}\right]^{\frac{n-i}{n}}\left[\frac{\psi_2(f_L(u)h_L^{n+1}(u))}{h_L^n(u)}\right]^{\frac{i}{n}}d\sigma(u).$$

If $\psi_1(t) = \psi_2(t) = t^{\frac{p}{n+p}}$ for $-n < p \le 0$, then $as_i(\psi_1,K;\psi_2,L)$ equals to the i-th mixed p-affine surface area for $-n < p \le 0$. On the other hand, if $\psi_1(t) = \psi_2(t) = t^{\frac{n}{n+p}}$ for $p < -n$, then $as_i^*(\psi_1,K;\psi_2,L)$ equals to the i-th mixed p-affine surface area for $p < -n$ [53].

Proposition 4.1 *If $j < i < k$ or $k < i < j$ (equivalently, $\frac{k-j}{k-i} > 1$), then for all $\phi_1, \phi_2 \in Conc(0,\infty)$ and $K, L \in C^2_+$,*

$$as_i(\phi_1, K; \phi_2, L) \leq as_j(\phi_1, K; \phi_2, L)^{\frac{k-i}{k-j}} as_k(\phi_1, K; \phi_2, L)^{\frac{i-j}{k-j}}.$$

Equality holds if (1) $K = L$ and $\phi_1 = \lambda\phi_2$ for some $\lambda > 0$; (2) ϕ_1 is homogeneous of degree $r \in [0,1)$, $\phi_1 = \lambda\phi_2$ for some $\lambda > 0$, and K and L are dilates of each other.

Remark. Similar results can be obtained for other general i-th mixed affine surface areas, for instance, $as_i(\psi_1, K; \psi_2, L) \leq as_j(\psi_1, K; \psi_2, L)^{\frac{k-i}{k-j}} as_k(\psi_1, K; \psi_2, L)^{\frac{i-j}{k-j}}$.

Proof. By formula (4.13), one has

$$\begin{aligned} as_i(\phi_1,K;\phi_2,L) &= \int_{S^{n-1}} \left[\phi_1\left(\frac{f_K^{-1}(u)}{h_K^{n+1}(u)}\right) h_K(u) f_K(u)\right]^{\frac{n-i}{n}} \left[\phi_2\left(\frac{f_L^{-1}(u)}{h_L^{n+1}(u)}\right) h_L(u) f_L(u)\right]^{\frac{i}{n}} d\sigma(u) \\ &= \int_{S^{n-1}} \left\{\left[\phi_1\left(\frac{f_K^{-1}(u)}{h_K^{n+1}(u)}\right) h_K(u) f_K(u)\right]^{\frac{n-j}{n}} \left[\phi_2\left(\frac{f_L^{-1}(u)}{h_L^{n+1}(u)}\right) h_L(u) f_L(u)\right]^{\frac{j}{n}}\right\}^{\frac{k-i}{k-j}} \\ &\quad \times \left\{\left[\phi_1\left(\frac{f_K^{-1}(u)}{h_K^{n+1}(u)}\right) h_K(u) f_K(u)\right]^{\frac{n-k}{n}} \left[\phi_2\left(\frac{f_L^{-1}(u)}{h_L^{n+1}(u)}\right) h_L(u) f_L(u)\right]^{\frac{k}{n}}\right\}^{\frac{i-j}{k-j}} d\sigma(u) \\ &\leq as_j(\phi_1, K; \phi_2, L)^{\frac{k-i}{k-j}} as_k(\phi_1, K; \phi_2, L)^{\frac{i-j}{k-j}}, \end{aligned}$$

where the last inequality follows Hölder inequality and formula (4.13). Clearly, the equality holds true if (1) $K = L$ and $\phi_1 = \lambda\phi_2$ for some $\lambda > 0$; (2) ϕ_1 is homogeneous of degree $r \in [0,1)$, $\phi_1 = \lambda\phi_2$ for some $\lambda > 0$, and K, L are dilates of each other.

If $j = 0, k = n$, then for all $0 \leq i \leq n$,

$$\big[as_i(\phi_1, K; \phi_2, L)\big]^n \leq [as_{\phi_1}(K)]^{n-i}[as_{\phi_2}(L)]^i. \tag{4.14}$$

On the other hand, if $i = 0$, $j = n$, $k \leq 0$, or $i = n, j = 0$, $k \geq n$, one has

$$\big[as_k(\phi_1, K; \phi_2, L)\big]^n \geq [as_{\phi_1}(K)]^{n-k}[as_{\phi_2}(L)]^k. \tag{4.15}$$

The following proposition gives the Santaló-type inequality for the general i-th mixed L_ϕ-affine surface area. Similar results for the general i-th mixed L^*_ϕ-affine surface area also hold.

Proposition 4.2 *Let $0 \le i \le n$, and $K,L \in C_+^2$ be convex bodies with centroid at the origin. For $\phi_1,\phi_2 \in Conc(0,\infty)$ with homogeneous degrees $r_1,r_2 \in [0,1)$ respectively,*

$$as_i(\phi_1, K; \phi_2, L) as_i(\phi_1, K^\circ; \phi_2, L^\circ) \le [as_i(\phi_1, B_2^n; \phi_2, B_2^n)]^2.$$

Equality holds true if K and L are ellipsoids that are dilates of one another.

Proof. By inequalities (3.10) and (4.14), one has

$$\begin{aligned} as_i(\phi_1,K;\phi_2,L)as_i(\phi_1,K^\circ;\phi_2,L^\circ) &\le \big[as_{\phi_1}(K)as_{\phi_1}(K^\circ)\big]^{\frac{n-i}{n}}\big[as_{\phi_2}(L)as_{\phi_2}(L^\circ)\big]^{\frac{i}{n}} \\ &\le n^2\big[|K||K^\circ|\phi_1(1)^2\big]^{\frac{n-i}{n}}\big[|L||L^\circ|\phi_2(1)^2\big]^{\frac{i}{n}} \\ &\le n^2\phi_1(1)^{\frac{2(n-i)}{n}}\phi_2(1)^{\frac{2i}{n}}|B_2^n|^2 \\ &= \big[as_i(\phi_1, B_2^n; \phi_2, B_2^n)\big]^2, \end{aligned}$$

where the last inequality follows from Blaschke-Santaló inequality and $0 \le i \le n$. Clearly, the equality holds true if K, L are ellipsoids that are dilates of one another.

The following proposition states the affine isoperimetric inequality for the general i-th mixed L_ϕ-affine surface area.

Proposition 4.3 *Let $0 \le i \le n$ and $K, L \in C_+^2$ be convex bodies with centroid at the origin. For $\phi_1, \phi_2 \in Conc(0,\infty)$, one has*

(i) $as_i(\phi_1, K; \phi_2, L) \le as_i(\phi_1, B_K; \phi_2, B_L)$, with equality if K and L are ellipsoids that are dilates of each another;

(ii) if in addition, ϕ_1, ϕ_2 are homogeneous of degrees $r_1, r_2 \in [0,1)$ respectively,

$$\left(\frac{as_i(\phi_1, K; \phi_2, L)}{as_i(\phi_1, B_2^n; \phi_2, B_2^n)}\right)^n \le \left(\frac{|K|}{|B_2^n|}\right)^{(n-i)(1-2r_1)} \left(\frac{|L|}{|B_2^n|}\right)^{i(1-2r_2)}.$$

Equality holds true if K and L are ellipsoids that are dilates of one another.

Remark. Similarly, one can get the affine isoperimetric inequality for the general i-th mixed L_ϕ^*-affine surface area. For instance, if $\phi_1, \phi_2 \in Conc(0,\infty)$ with homogeneous degrees $r_1, r_2 \in [0,1)$ respectively, then

$$\left(\frac{as_i^*(\phi_1, K; \phi_2, L)}{as_i^*(\phi_1, B_2^n; \phi_2, B_2^n)}\right)^n \le \left(\frac{|K|}{|B_2^n|}\right)^{(n-i)(2r_1-1)} \left(\frac{|L|}{|B_2^n|}\right)^{i(2r_2-1)}.$$

Equality holds if K and L are ellipsoids that are dilates of one another.

Proof. (i) Note $[as_i(\phi_1, B_K; \phi_2, B_L)]^n = [as_{\phi_1}(B_K)]^{n-i}[as_{\phi_2}(B_L)]^i$. The desired result is then an immediate consequence of inequality (4.14), and Ludwig's isoperimetic inequality [24].

(ii) By inequality (4.14), and $[as_i(\phi_1, B_2^n; \phi_2, B_2^n)]^n = [as_{\phi_1}(B_2^n)]^{n-i}[as_{\phi_2}(B_2^n)]^i$,

$$\left[\frac{as_i(\phi_1, K; \phi_2, L)}{as_i(\phi_1, B_2^n; \phi_2, B_2^n)}\right]^n \quad \leq \quad \left[\frac{as_{\phi_1}(K)}{as_{\phi_1}(B_2^n)}\right]^{n-i}\left[\frac{as_{\phi_2}(L)}{as_{\phi_2}(B_2^n)}\right]^i.$$

Combining with inequality (3.12) and $r_1, r_2 \in [0, 1)$, the desired result follows. Clearly, equality holds if K and L are ellipsoids that are dilates of one another.

Proposition 4.4 *Let $K \in C_+^2$ be a convex body with centroid at the origin. For $k \geq n$, and $\phi_1, \phi_2 \in Conc(0, \infty)$, one has*

(i) $as_k(\phi_1,K;\phi_2,B_2^n) \geq as_k(\phi_1,B_K;\phi_2,B_2^n)$, with equality if K is a ball;

(ii) if in addition, ϕ_1 is homogeneous of degree $r_1 \in [0, 1)$, then

$$\left(\frac{as_k(\phi_1, K; \phi_2, B_2^n)}{as_k(\phi_1, B_2^n; \phi_2, B_2^n)}\right)^n \geq \left(\frac{|K|}{|B_2^n|}\right)^{(n-k)(1-2r_1)},$$

with equality if K is a ball; Moreover,

$$as_k(\phi_1, K; \phi_2, B_2^n)as_k(\phi_1, K^\circ; \phi_2, B_2^n) \geq [as_k(\phi_1, B_2^n; \phi_2, B_2^n)]^2,$$

with equality if K is a ball.

Proof. (i) By inequality (4.15) and $[as_k(\phi_1,B_K;\phi_2,B_2^n)]^n = [as_{\phi_1}(B_K)]^{n-k}[as_{\phi_2}(B_2^n)]^k$,

$$\left[\frac{as_k(\phi_1,K;\phi_2,B_2^n)}{as_k(\phi_1,B_K;\phi_2,B_2^n)}\right]^n \quad \geq \quad \left[\frac{as_{\phi_1}(K)}{as_{\phi_1}(B_K)}\right]^{n-k} \geq 1,$$

where we have used the Ludwig's isoperimetric inequality in [24] and $n - k \leq 0$. The equality holds trivially if K is a ball.

(ii) Again, by inequality (4.15) and $[as_k(\phi_1, B_2^n; \phi_2, B_2^n)]^n = [as_{\phi_1}(B_2^n)]^{n-k}[as_{\phi_2}(B_2^n)]^k$,

$$\left[\frac{as_k(\phi_1, K; \phi_2, B_2^n)}{as_k(\phi_1, B_2^n; \phi_2, B_2^n)}\right]^n \quad \geq \quad \left[\frac{as_{\phi_1}(K)}{as_{\phi_1}(B_2^n)}\right]^{n-k} \geq \left(\frac{|K|}{|B_2^n|}\right)^{(n-k)(1-2r_1)},$$

where the last inequality follows inequality (3.12) and $n - k \leq 0$. Clearly if K is a ball, the equality holds.

Theorem 3.1 implies that $as_{\phi_1}(K)as_{\phi_1}(K^\circ) \le [as_{\phi_1}(B_2^n)]^2$. Combining with inequality (4.15) and $n-k \le 0$, we have

$$\begin{aligned} as_k(\phi_1, K; \phi_2, B_2^n)as_k(\phi_1, K^\circ; \phi_2, B_2^n) &\ge [as_{\phi_1}(K)as_{\phi_1}(K^\circ)]^{\frac{n-k}{n}}[as_{\phi_2}(B_2^n)]^{\frac{2k}{n}} \\ &\ge [as_{\phi_1}(B_2^n)]^{\frac{2(n-k)}{n}}[as_{\phi_2}(B_2^n)]^{\frac{2k}{n}} \\ &= [as_k(\phi_1, B_2^n; \phi_2, B_2^n)]^2. \end{aligned}$$

Clearly if K is a ball, equality holds true.

Proposition 4.5 *Let $K \in C_+^2$ be a convex body with centroid at the origin. For $k \le 0$, and $\psi_1, \psi_2 \in Conv(0,\infty)$, one has*

(i) $as_k(\psi_1,K;\psi_2,B_2^n) \ge as_k(\psi_1,B_K;\psi_2,B_2^n)$, with equality if K is a ball;

(ii) if in addition, ψ_1 is homogeneous of degree $r_1 \in (-\infty, 0]$, then

$$\left(\frac{as_k(\psi_1, K; \psi_2, B_2^n)}{as_k(\psi_1, B_2^n; \psi_2, B_2^n)}\right)^n \ge \left(\frac{|K|}{|B_2^n|}\right)^{(n-k)(1-2r_1)},$$

with equality if K is a ball; Moreover,

$$as_k(\psi_1, K; \psi_2, B_2^n)as_k(\psi_1, K^\circ; \psi_2, B_2^n) \ge c^{n-k}[as_k(\psi_1, B_2^n; \psi_2, B_2^n)]^2,$$

where c is the universal constant in the inverse Santaló inequality [7, 22, 38, 39, 41]; namely, $|K||K^\circ| \ge c^n|B_2^n|^2$.

Proof. (i) For the L_ψ-affine surface area with $\psi \in Conv(0,\infty)$, Ludwig proved the affine isoperimetric inequality [24]: $as_\psi(K) \ge as_\psi(B_K)$ with equality if and only if K is an ellipsoid. If $\psi \in Conv(0,\infty)$ is homogeneous of degree $r \in (-\infty, 0]$, then

$$\frac{as_\psi(K)}{as_\psi(B_2^n)} \ge \left(\frac{|K|}{|B_2^n|}\right)^{1-2r}, \tag{4.16}$$

with equality if and only if K is an ellipsoid.

Similar to inequality (4.15), one can prove that, for $k \le 0$,

$$\big[as_k(\psi_1, K; \psi_2, L)\big]^n \ge [as_{\psi_1}(K)]^{n-k}[as_{\psi_2}(L)]^k. \tag{4.17}$$

Combining with $[as_k(\psi_1, B_K; \psi_2, B_2^n)]^n = [as_{\psi_1}(B_K)]^{n-k}[as_{\psi_2}(B_2^n)]^k$, one has

$$\left[\frac{as_k(\psi_1, K; \psi_2, B_2^n)}{as_k(\psi_1, B_K; \psi_2, B_2^n)}\right]^n \ge \left[\frac{as_{\psi_1}(K)}{as_{\psi_1}(B_K)}\right]^{n-k} \ge 1,$$

where we have used Ludwig's isoperimetric inequality in [24] and $k \le 0$. Equality holds trivially if K is a ball.

(ii) Again by inequality (4.17) and $[as_k(\psi_1, B_2^n; \psi_2, B_2^n)]^n = [as_{\psi_1}(B_2^n)]^{n-k}[as_{\psi_2}(B_2^n)]^k$,

$$\left[\frac{as_k(\psi_1, K; \psi_2, B_2^n)}{as_k(\psi_1, B_2^n; \psi_2, B_2^n)}\right]^n \quad \ge \quad \left[\frac{as_{\psi_1}(K)}{as_{\psi_1}(B_2^n)}\right]^{n-k} \ge \left(\frac{|K|}{|B_2^n|}\right)^{(n-k)(1-2r_1)},$$

where the last inequality follows inequality (4.16) and $n-k \ge 0$. Clearly if K is a ball, the equality holds.

The following inequality was proved in [24]: if $\psi \in Conv(0,\infty)$, then

$$as_\psi(K) \ge n|K|\psi\left(\frac{|K^\circ|}{|K|}\right).$$

(Note that it holds true for all $K \in C_+^2$). The inverse Santaló inequality says that $|K||K^\circ| \ge c^n|B_2^n|^2$, where c is a universal constant [7, 22, 38, 39, 41]. Therefore

$$as_\psi(K)as_\psi(K^\circ) \ge n^2|K||K^\circ|\psi(1)^2 \ge c^n\psi(1)^2n^2|B_2^n|^2 = c^n\left[as_\psi(B_2^n)\right]^2.$$

Combining with inequality (4.17) and $n-k \ge 0$,

$$\begin{aligned} as_k(\psi_1, K; \psi_2, B_2^n)as_k(\psi_1, K^\circ; \psi_2, B_2^n) &\ge [as_{\psi_1}(K)as_{\psi_1}(K^\circ)]^{\frac{n-k}{n}}[as_{\psi_2}(B_2^n)]^{\frac{2k}{n}} \\ &\ge c^{n-k}[as_{\psi_1}(B_2^n)]^{\frac{2(n-k)}{n}}[as_{\psi_2}(B_2^n)]^{\frac{2k}{n}} \\ &= c^{n-k}[as_k(\psi_1, B_2^n; \psi_2, B_2^n)]^2. \end{aligned}$$

Proposition 4.6 *Let $K \in C_+^2$ be a convex body with centroid at the origin. For $k \le 0$, and $\psi_1, \psi_2 \in Conv(0,\infty)$, one has*

(i) $as_k^*(\psi_1,K;\psi_2,B_2^n) \ge as_k^*(\psi_1,(B_{K^\circ})^\circ;\psi_2,B_2^n)$, with equality if K is a ball;

(ii) if in addition, ψ_1 is homogeneous of degree $r_1 \in (-\infty, 0]$, one has

$$\left(\frac{as_k^*(\psi_1, K; \psi_2, B_2^n)}{as_k^*(\psi_1, B_2^n; \psi_2, B_2^n)}\right)^n \ge c^{n(1-2r_1)(n-k)}\left(\frac{|K|}{|B_2^n|}\right)^{(n-k)(2r_1-1)},$$

Moreover, $as_k^*(\psi_1, K; \psi_2, B_2^n)as_k^*(\psi_1, K^\circ; \psi_2, B_2^n) \ge c^{n-k}[as_k^*(\psi_1, B_2^n; \psi_2, B_2^n)]^2$, where c is the same constant as in Proposition 4.5.

Proof. (i) Recall that $as_\psi^*(K) = as_\psi(K^\circ)$ (see Proposition 2.1). By inequality (4.16), and Ludwig's isoperimetric inequality in [24],

$$\left[\frac{as_k^*(\psi_1,K;\psi_2,B_2^n)}{as_k^*(\psi_1,(B_{K^\circ})^\circ;\psi_2,B_2^n)}\right]^n \ge \left[\frac{as_{\psi_1}^*(K)}{as_{\psi_1}^*((B_{K^\circ})^\circ)}\right]^{n-k} = \left[\frac{as_{\psi_1}(K^\circ)}{as_{\psi_1}(B_{K^\circ})}\right]^{n-k} \ge 1.$$

Clearly, the equality holds if K is a ball.

(ii) Again, by inequality (4.16), the inverse Santaló inequality and $r_1 \in (-\infty, 0]$,

$$\frac{as^*_\psi(K)}{as^*_\psi(B^n_2)} = \frac{as_\psi(K^\circ)}{as_\psi(B^n_2)} \geq \left(\frac{|K^\circ|}{|B^n_2|}\right)^{1-2r_1} \geq c^{n(1-2r_1)} \left(\frac{|K|}{|B^n_2|}\right)^{2r_1-1}. \tag{4.18}$$

Similar to inequality (4.15), one can prove that for $k \leq 0$,

$$\big[as^*_k(\psi_1, K; \psi_2, L)\big]^n \geq [as^*_{\psi_1}(K)]^{n-k}[as^*_{\psi_2}(L)]^k.$$

As $n-k \geq 0$ and inequality (4.18), one has

$$\left[\frac{as^*_k(\psi_1, K; \psi_2, B^n_2)}{as^*_k(\psi_1, B^n_2; \psi_2, B^n_2)}\right]^n \quad \geq \quad \left[\frac{as^*_{\psi_1}(K)}{as^*_{\psi_1}(B^n_2)}\right]^{n-k} \geq c^{n(1-2r_1)(n-k)} \left(\frac{|K|}{|B^n_2|}\right)^{(2r_1-1)(n-k)}.$$

The proof of $as_k(\psi_1, K; \psi_2, B^n_2) as_k(\psi_1, K^\circ; \psi_2, B^n_2) \geq c^{n-k}[as_k(\psi_1, B^n_2; \psi_2, B^n_2)]^2$ is same as that of Proposition 4.5.

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